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Note

What Can Be Said about Pure O-Sequences?*

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Let $f(\Delta)=(f_0, f_1, ..., f_{d-1})$ be the f-vector of a Cohen–Macaulay complex Δ. Björner proved that (*) $f_i \leqslant f_{(d-2)-i}$ for any $0 \leqslant i < [d/2]$ and (**) $f_0 \leqslant f_1 \leqslant \cdots \leqslant f_{[(d-1)/2]}$. Recently, Stanley generalized Björner's inequalities (*) and (**) for pure simplicial complexes. In this paper we consider O-sequence analogue of the inequalities (*) and (**). Let $(h_0, h_1, ..., h_s)$, $h_s \neq 0$, is a pure O-sequence. We shall prove that $h_i \leqslant h_{s-i}$ for any $0 \leqslant i \leqslant [s/2]$ and $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{[(s+1)/2]}$. © 1989 Academic Press, Inc.

INTRODUCTION

In 1927, Macaulay [5] studied Hilbert functions of standard G-algebras (in the terminology of [7]) and obtained an explicit numerical characterization of O-sequences, see [7, (2.2)]. On the other hand, Kruskal and later, independently, Katona gave an answer to the problem of characterizing the face vectors of simplicial complexes. Consult [1] for further information.

Let $f(\Delta)=(f_0, f_1, ..., f_{d-1})$ be the f-vector of a $(d-1)$-dimensional simplicial complex Δ. Define the h-vector $h(\Delta)=(h_0, h_1, ..., h_d)$ by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i},$$

where we set $f_{-1}=1$. Stanley [6] found a complete characterization of the h-vectors of Cohen–Macaulay complexes. On the other hand, Björner [2] proved the face number inequalities (*) $f_i \leqslant f_{(d-2)-i}$ for any i,

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$0 \leqslant i < [d/2]$, and (**) $f_0 \leqslant f_1 \leqslant \cdots \leqslant f_{[(d-1)/2]}$ for Cohen–Macaulay complexes. Björner's proof depends on the fact that the h-vector of a Cohen–Macaulay complex is non-negative, i.e., $h_i \geqslant 0$ for any i. Recently, Stanley extended Björner's inequalities (*) and (**) for *any* pure simplicial complex. The Kruskal–Katona theorem is essentially used in the proof of Stanley.

Now, the purpose of this paper is to generalize the inequalities (*) and (**) for pure O-sequences. Our result (1.1) is a starting point toward the problem of finding a characterization of pure O-sequences.

The author thanks Professor Richard P. Stanley for suggesting the key lemma (2.1) and Professor Anders Björner for sending the interesting paper [2].

1. THE INEQUALITY $h_i \leqslant h_{s-i}$

Let $Y_1, Y_2, ..., Y_v$ be indeterminates. A non-empty set $\mathscr{M}$ of monomials $Y_1^{a_1} Y_2^{a_2} \cdots Y_v^{a_v}$ in the variables $Y_1, Y_2, ..., Y_v$ is said to be an *order ideal of monomials* if, whenever $M \in \mathscr{M}$ and N divides M, then $N \in \mathscr{M}$. Equivalently, if $Y_1^{a_1} Y_2^{a_2} \cdots Y_v^{a_v} \in \mathscr{M}$ and $0 \leqslant b_i \leqslant a_i$, then $Y_1^{b_1} Y_2^{b_2} \cdots Y_v^{b_v} \in \mathscr{M}$. In particular, since $\mathscr{M}$ is assumed non-empty, $1 \in \mathscr{M}$.

A finite or infinite sequence $(h_0, h_1, ...)$ of non-negative integers is said to be an *O-sequence* if there exists an order ideal $\mathscr{M}$ of monomials in variables $Y_1, Y_2, ..., Y_v$ with each $\deg Y_i = 1$ such that $h_j = \#\{M \in \mathscr{M}; \deg M = j\}$ for any $j = 0, 1, ...$. In particular, $h_0 = 1$. A finite order ideal $\mathscr{M}$ of monomials is called *pure* if the maximal elements of $\mathscr{M}$ (ordered by divisibility) all have the same degree. We define a *pure O-sequence* in the obvious way.

What can be said about pure O-sequences?

(1.1) THEOREM. *Suppose that an O-sequence $(h_0, h_1, ..., h_s)$, $h_s \neq 0$, is pure. Then we have the inequality $h_i \leqslant h_{s-i}$ for any i, $0 \leqslant i \leqslant [s/2]$.*

It would, of course, be of great interest to find a characterization of pure O-sequences.

(1.2) EXAMPLE. A finite sequence (h_0, h_1, h_2) of positive integers is a pure O-sequence if and only if $h_0 = 1$ and

$$[(h_1+1)/2] \leqslant h_2 \leqslant \binom{h_1+1}{h_1-1}.$$

On the other hand, if $(h_0, h_1, ..., h_s)$ with $h_s \neq 0$ is a pure O-sequence then the O-sequence $(h_0, h_1, ..., h_t)$ is also pure for any t, $0 \leqslant t \leqslant s$. Hence, as a curious consequence of (1.1), we obtain

(1.3) COROLLARY. *Let* $(h_0, h_1, ..., h_s)$ *with* $h_s \neq 0$ *be a pure O-sequence. Then we have the inequalities* $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{[(s+1)/2]}$.

2. A PROOF OF (1.1)

Let $Y_1, Y_2, ..., Y_v$ be indeterminates with each $\deg Y_i = 1$ and M a monomial $Y_1^{a_1} Y_2^{a_2} \cdots Y_v^{a_v}$ with $a_1 + a_2 + \cdots + a_v = n$. Write $\binom{M}{i}$ for the set of monomials N of degree i such that N divides M. Note that $\#\binom{M}{i} = \#\binom{M}{n-i}$ for any i, $0 \leqslant i \leqslant [n/2]$. Also, let $\mathscr{D}_n^{(M)}$ be the divisor lattice $\bigcup_{0 \leqslant i \leqslant n} \binom{M}{i}$ ordered by divisibility. In particular, if M is square-free then $\mathscr{D}_n^{(M)}$ coincides with the Boolean lattice $\mathscr{B}_n$.

Now, DeBruijn *et al.* [3] guarantees

(2.1) LEMMA. *The lattice* $\mathscr{D}_n^{(M)}$ *can be partitioned into symmetric saturated chains, that is to say,* $\mathscr{D}_n^{(M)}$ *is a disjoint union of chains of the form* $\alpha_i < \alpha_{i+1} < \cdots < \alpha_{n-i}$ *such that* $\alpha_j \in \binom{M}{j}$ *for any* j, $i \leqslant j \leqslant n-i$.

Consult Greene and Kleitman [4] for further information. Also, from the above decomposition of $\mathscr{D}_n^{(M)}$, we immediately obtain

(2.2) COROLLARY. *For any* i, $0 \leqslant i \leqslant [n/2]$, *there exists a bijection* $\psi_{n,M}^{(i)}: \binom{M}{i} \to \binom{M}{n-i}$ *such that* N *divides* $\psi_{n,M}^{(i)}(N)$ *for every* $N \in \binom{M}{i}$.

We are now in the position to give a proof of (1.1). Let $(h_0, h_1, ..., h_s)$ with $h_s \neq 0$ be the pure O-sequence associated with a pure order ideal $\mathscr{M}$ of monomials. We shall prove the inequality $h_i \leqslant h_{s-i}$ by induction on h_s, i.e., the number of maximal elements of $\mathscr{M}$. Let $M_1, M_2, ..., M_{h_s}$ be the set of maximal elements of $\mathscr{M}$ and $\mathscr{N}$ the pure order ideal of monomials consisting of all monomials N such that N divides M_i for some i, $1 \leqslant i < h_s$. Write $h_i' = \#\{N \in \mathscr{N}; \deg N = i\}$. Then $(h_0', h_1', ..., h_s')$ is a pure O-sequence with $h_s' = h_s - 1$. Hence, by assumption of induction, we obtain $h_i' \leqslant h_{s-i}'$ for any i, $0 \leqslant i \leqslant [s/2]$. Now, suppose that $N \in \binom{M_{h_s}}{i}$ does not belong to $\mathscr{N}$. Then, $\psi_{s,M_{h_s}}^{(i)}(N) \in \binom{M_{h_s}}{s-i}$ is not contained in $\mathscr{N}$ since N divides $\psi_{s,M_{h_s}}^{(i)}(N)$. Hence, $h_i - h_i' \leqslant h_{s-i} - h_{s-i}'$ since $\psi_{s,M_{h_s}}^{(i)}$ is bijective. Thus we obtain the inequality $h_i \leqslant h_{s-i}$ as desired. This completes our proof of (1.1). Q.E.D.

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